Two hundred and ninety-fourth Meeting.

April 6, 1847. — MONTHLY MEETING.

The PRESIDENT in the chair.

Professor Strong, of New Brunswick, New Jersey, communicated the following papers, viz.: —

I. "*An attempt to prove that the sum of the three angles of any rectilineal triangle is equal to two right angles.*

"*Def.* Two quantities are said to be of the same kind, when the less can be multiplied by some positive integer, so as to exceed the greater. Thus, if A and B are quantities of the same kind, and if A is greater than B, then some positive integer, m, may be found, such that the inequality $m\,B > A$ shall exist. For if m is taken greater than the quotient arising from the division of A by B, then evidently there results the inequality $m\,B > A$, as required.

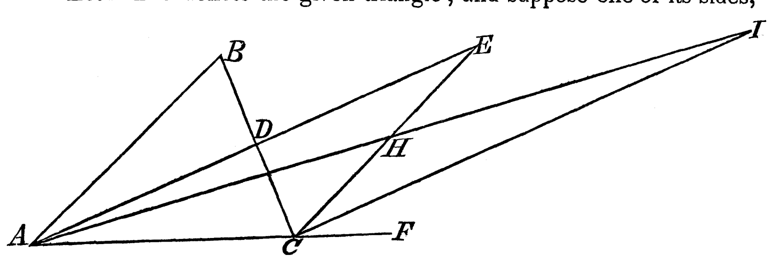
"*Dem.* If m denotes any positive integer, then shall the inequality $2^m > m$ obtain. For the first member of the inequality denotes the product arising from taking 2 as a factor as often as these units in m, whereas the second member is the sum of the units represented by m, and the inequality is evident.

"*Cor.* If we take A and B, as above, and $m\,B > A$, there results $B > \frac{A}{m}$; much more, then, shall the inequality $B > \frac{A}{2^m}$ have place. This follows at once since it has been shown that 2^m is greater than m; and it is evident that if m is an indefinitely great number, $\frac{A}{m}$ is indefinitely greater than $\frac{A}{2^m}$.

"*Ax.* No angle of a rectilineal triangle can exceed two right angles.

"*Prop.* 1. To find a triangle that shall have the sum of its angles equal to the sum of the angles of any given triangle.

"Let $A\,B\,C$ denote the given triangle; and suppose one of its sides,



$B\,C$, is bisected at D, and that D and the opposite angle A are connected by the right line $A\,D$, which is produced in the direction $A\,D$

to E, so that $D E = A D$, and that the point E and the angle C are connected by a right line; then shall the sum of the angles of the triangle $A C E$ equal that of the triangle $A B C$.

"For the opposite vertical angles $A D B$, $C D E$ are equal (Simson's Euclid, B. I., prop. 15), and by construction $B D = D C$, $A D = D E$; hence (Sim., B. I., p. 4) the triangles $A D B$, $C D E$ are identical; so that their bases $A B$, $C E$ are equal, and their angles $A B D$, $E C D$ are equal; also the angle $B A D$ equals the angle $C E D$. Hence the angle C of the triangle $A C E$ equals the sum of the two angles B and C of the given triangle ($A B C$), and the sum of the angles A and E of the triangle $A C E$ is equal to the angle A of the given triangle ($A B C$); $\therefore$ the sum of the angles of the triangle $A C E$ is equal to that of the given triangle $A B C$, as required.

"*Cor.* 1. Let the angle $B A C$ of the given triangle be denoted by A; then if $C E$ ($= A B$) is not greater than $A C$, the angle $C A E$ is not greater than $C E A$ (Sim., props. 5, 19, B. I.); hence the angle A of the triangle $A C E$ is not greater than $\frac{A}{2}$; we shall call the triangle $A C E$ the first derived triangle. Of the two sides $A C$ and $C E$ of the triangle $A C E$, let $C E$ be that which is not the greater; and let a right line be drawn from the angle A, opposite to the side $C E$, through the point, H, of bisection of $C E$, and suppose the line thus drawn to be produced in the direction $A H$ to I, so that $H I = H A$; then connect the point I and the angle C of the triangle $A C E$ by a right line; and there will be formed the triangle $A C I$. In the same way that it was shown that the sum of the angles of the triangle $A C E$ is equal to the sum of the angles of the (given) triangle $A B C$, it may be shown that the sum of the angles of the triangle $A C I$ is equal to that of $A C E$; consequently the sum of the angles of $A C I$ equals that of the given triangle $A B C$. And if $A C$ is not greater than $C I$, then it may be shown (as before) that the angle I is not greater than the angle $C A E \div 2$, and since $C A E$ is not greater than $\frac{A}{2}$, $\therefore$ the angle I is not greater than $\frac{A}{2^2}$, we shall call $A C I$ the second derived triangle.

"We may in the same way (that we derived the triangle $A C I$ from $A C E$) derive a triangle from $A C I$ (called the third derived triangle), having the sum of its angles equal to that of $A C I$, and of course equal to that of the given triangle $A B C$, and having one of its angles not greater than the angle $A I C \div 2$, and consequently not greater

than $\frac{A}{2^3}$. And proceeding in the same way from triangle to triangle, until we obtain the m^{th} derived triangle, then the sum of its angles will equal that of the given triangle $A\,B\,C$, and one of its angles will not be greater than $\frac{A}{2^m}$; where m is of course a positive integer.

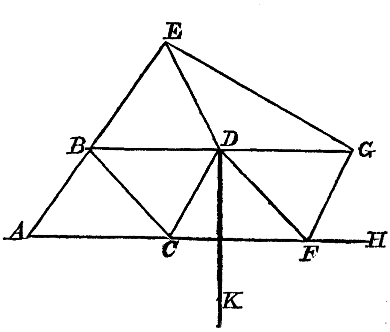
"*Cor.* 2. If we obtain the derived triangle whose number is $m+1$, the sum of two of its angles will equal that angle of the m^{th} triangle which has been shown not to be greater than $\frac{A}{2^m}$; we hence see how from any given triangle to derive another triangle such that the sum of its angles shall equal that of the given triangle, and such that the sum of two of its angles shall not be greater than $\frac{A}{2^m}$; where A denotes one of the angles of the given triangle.

" *Remark.* Cor. 1 is substantially the same as Mr. I. Ivory's process, given at page 189 of the New York edition of J. R. Young's *Elements of Geometry.*

" *Prop.* 2. The sum of the angles of any triangle is not greater than two right angles.

" Let the triangle $A\,B\,C$, of Prop. 1, represent any triangle, and denote a right angle by R, and if possible let the sum of the angles of the triangle equal $2\,A + V$, V being a finite positive angle. Then, using A to represent the angle $B\,A\,C$ of the triangle, some positive integer, m, may be found so that the inequality $m\,V > A$ shall exist. From $m\,V > A$, it follows that $V > \frac{A}{m} > \frac{A}{2^m}$, or V is greater than $\frac{A}{2^m}$. By Cor. 2, Prop. 1, we may derive a triangle from $A\,B\,C$, such that the sum of two of its angles shall not be greater than $\frac{A}{2^m}$; hence the sum of these two angles is less than V; consequently the third angle of the triangle must be greater than $2\,R$, which is impossible. Hence the sum of the angles of the triangle $A\,B\,C$ is not greater than two right angles.

" *Prop.* 3. The sum of the angles of any triangle is greater than a right angle.



" Let $A\,B\,C$ represent any triangle, and suppose (for convenience) that the angle $A\,B\,C$ is not less than either of the other angles of the triangle. Let the base be extended in the direction $A\,C$ to F, so that $C\,F$ equals the base ($A\,C$), and through C and F draw the right lines $C\,D$ and $F\,G$, each equal to $A\,B$, and so

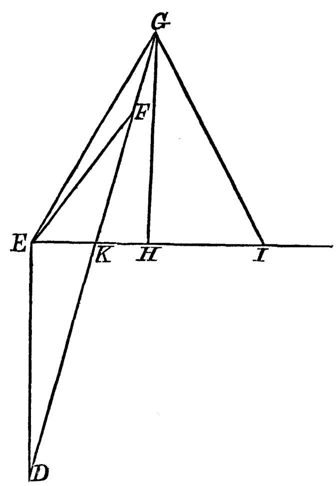
as to make the angles DCF, GFH each equal to the angle BAC, and connect the points D and F, D and G, D and B, by right lines; also through D draw DK, at right angles to DG. Since $AB = DC$, $AC = CF$, and the angle BAC equals the angle DCF, the triangles ABC, CDF are identical (Sim., B. I., p. 4), making the sides DF and BC equal to each other, and the angle CDF equal to the angle ABC, and the angle CFD equal to the angle ACB; hence the sum of the angles BAC, BCA equals the sum of the angles BCA, DCF, which equals the sum of the angles CFD, GFH. Since the sum of the three angles at C makes two right angles, and that the sum of the angles at F makes two right angles (Sim., B. I., p. 13), it follows from what has been proved that the angles BCD, DFG are equal, and since $BC = DF$, $CD = FG$, the triangles BCD, DFG are identical (Sim., B. I., p. 4), making BD equal to DG, and the angle CBD equal to the angle FDG, and the angle CDB equal to the angle FGD; hence the sum of the angles CBD, CDB equals the sum of the angles CDB, FDG. By Prop. 2, since the sum of the three angles of any triangle is not greater than two right angles, and that the three angles at C make two right angles, it follows that the sum of the angles B and D of the triangle BCD is not greater than the sum of the angles ACB, FCD; hence, and from what has been proved, it follows that the sum of the angles of the triangle ABC is not less than the sum of the angles BDC, CDF, FDG; but it is evident that the sum of these angles exceeds the right angle KDG by a fine angle; hence the sum of the three angles of the triangle ABC exceeds a right angle, as required.

"*Remark.* If AB is extended in the direction AB to E, so that $BE = BD$, and if the points D and E, G and E, are connected by right lines, the point D falls evidently within the pentagonal figure $CBEGF$; and if R denotes a right angle, the sum of the angles BDC, CDF, FDG, GDE, EDB is equal to $4R$ (Sim., B. I., p. 15, Cor. 2). Since the sum of the angles ABD, DBE is equal to $2R$, the sum of the angles at the base of the isosceles triangle BDE is not greater than the angle ABD; consequently the angle BDE is not greater than the angle $ABD \div 2$, which is not greater than half the sum of the angles BDC, CDF, FDG. We now observe that the sum of the angles of the triangle ABC is not less than $R + \frac{5}{16}R$. For if the sum of the angles of the triangle ABC is not greater than $R + \frac{5}{16}R$, then by what has been shown the sum of the angles BDC,

$C D F$, $F D G$, $B D E$ is not greater than $R + \frac{5}{16} R + \frac{R}{2} + \frac{5}{32} R = \frac{63}{32} R$; consequently the angle D of the triangle $E D G$ is not less than $4 R - \frac{63}{32} R = \frac{65}{32} R = 2 R + \frac{R}{32}$, which is impossible; hence the sum of the angles of any triangle, $A B C$, is greater than $R + \frac{5}{16} R$, as required.

" *Prop.* 4. The sum of the angles of any triangle is not less than two right angles.

" If the sum of the angles of any one triangle is not the same as that of any other, then there are evidently some triangles having the sum of their angles less than that of any others; let, therefore, $A B C$ (see fig., Prop. 1) denote a triangle such that the sum of its angles is not greater than that of any other triangle. If R represents a right angle, then, since the sum of the angles of any triangle is greater than R, the sum of the angles of the triangle $A B C$ may be expressed by $R + V$, V being a positive angle. If we denote the angle $B A C$ of the given triangle by A, then, as in Prop. 2, we may find some positive integer, m, such that the inequality $V > \frac{A}{2^m}$ shall have place; and by Cor. 1 and 2 of Prop. 1 we may derive from the triangle $A B C$ another triangle represented by $D E F$ such that the sum of its angles shall equal that of $A B C$, and consequently equal $R + V$, and further



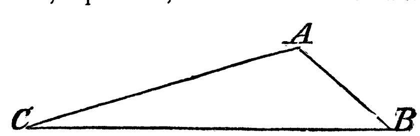
such that the sum of two of its angles, $E D F$, $E F D$, shall not be greater than $\frac{A}{2^m}$; $\therefore$ the sum of these (two) angles is less than V, consequently the third angle E of the triangle is greater than R. Of the two sides $D E$, $E F$, let $D E$ be that which is not the less, and through E draw $E K$, at right angles to $D E$; then, since the angle $D E F$ is greater than R, the perpendicular will of course meet the side $D F$ at some point, as K, between D and F. Since the sum of the sides $D E$ and $E F$ is greater than $D F$ (Sim., p. 20, B. I.), and that $D K$ is greater than $D E$ (Sim., p. 19, B. I.), we of course have $D K$ greater than $K F$. Hence extend $D F$ to G, so that $G K = D K$, and extend $E K$ to H, so that $K H = E K$, and connect G and H

by a right line; also extend $E H$ to I, so that $I H = E H$, and draw a right line from I to G. Since $D K = G K$, $E K = H K$, and the angles $D K E$, $G K H$ are equal (Sim., Prop. 15, B. I.), the triangles $D K E$, $G K H$ are identical (Sim., p. 4, B. I.), and $D E = G H$, the angle $G H K =$ the angle $D E K = R$, and the angle $H G K =$ the angle $E D K$; hence the triangles $E H G$, $I H G$ are identical, since they have $E H = I H$, $H G$ common, and that the angles at H are right (Sim., p. 4, B. I.); hence the sides $G E$, $G I$ are equal, the angle $G I H$ equals the angle $G E H$, and the angle $I G H$ equals the angle $E G H$. Since the angle $D F E$ is greater than either of the angles $F G E$, $F E G$ (Sim., p. 16, B. I.), it follows from what has been shown that the angle $E G H$ is not greater than the sum of the angles $E D F$, $E F D$, and of course the sum of the angles $E G H$, $K E F$ is not greater than V, and since $G E F$ is less than $E F D$, $G E F$ is not greater than $\frac{A}{2^m}$, $\therefore$ the sum of the angles $E G H$, $H E G$ is not greater than $V + \frac{A}{2^m}$, consequently the sum of the angles of the triangle $E G I$ is not greater than $2 V + \frac{2A}{2^m}$. But by hypothesis the sum of the angles of the triangle $A B C$, which equals the sum of the angles of the triangle $D E F$, is not greater than the sum of the angles of the triangle $E G I$; $\therefore 2 V + \frac{2A}{2^m}$ is not less than $R + V$, or V is not less than $R - \frac{2A}{2^m}$. Hence V cannot differ from R by any given angle, as a, so that $V = R - a$, a being a positive finite angle; for by taking a sufficiently great positive integer for m (which is evidently arbitrary), we shall make $\frac{2A}{2^m}$ less than a, which is absurd; $\therefore V$ is not less than R. Hence the sum of the angles of the triangle $A B C$ is not less than $2 R$.

" *Cor.* Since by Prop. 2 the sum of the angles of any triangle is not greater than $2 R$, and from what has been shown in this Prop. it is not less than $2 R$, it follows that the sum of the angles of any triangle $= 2 R =$ two right angles, as required.

" *Appendix to Propositions* 3 *and* 4.

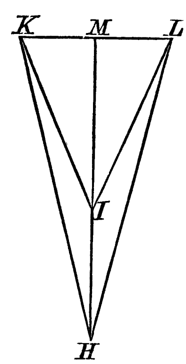
" *Lem.* No triangle can exist such that the sum of its angles shall be less than any given angle; or such that the sum of its angles shall equal an infinitesimal angle. For, if possible, let $A B C$ be such a triangle; then, since the sum of its angles is less than any given angle, each of its angles is of course less than



any given angle. Hence, since the angles C and B are infinitesimal angles, the sides $A\,C$ and $A\,B$ must coincide very nearly with the side $C\,B$, and $\therefore$ since $A\,C$ and $A\,B$ lie in opposite directions they cannot possibly come near to coincidence with each other; but since the angle A is less than any given angle (or infinitesimal), the sides $A\,C$, $A\,B$ must very nearly coincide with each other and have nearly the same direction, which is absurd. Hence the sum of the angles of a triangle is not less than any given angle (or infinitesimal), but it is a finite quantity, being equal to some finite angle, or the sum of finite angles.

"*Remark.* By aid of this lemma we are prepared to give a very simple demonstration of Prop. 3.

"*Prop.* 3′. The sum of the angles of any triangle is greater than a right angle. Let the triangle $A\,B\,C$, of Prop. 1, denote any triangle, and denote the angle $B\,A\,C$ by A, and use V to represent any small finite angle; then we may find some positive integer, m, such that the inequality $m\,\frac{V}{2} > A$ shall exist, consequently the inequality $\frac{V}{2} > \frac{A}{2^m}$ has place also. Hence, by Cor. 1 and 2 of Prop. 1, we can find a derived triangle, which we shall represent by the triangle $H\,I\,K$, such that the sum of its angles equals that of $A\,B\,C$, and the sum of two of its angles, $I\,H\,K$, $I\,K\,H$, is not greater than $\frac{A}{2^m}$, $\therefore$ $\frac{V}{2}$ is greater than the sum of these two angles. At the point I make the angle $H\,I\,L$, equal to the angle $H\,I\,K$;



also make the right line $I\,L$ equal to the side $I\,K$, and draw a right line from H to L; then (Sim., p. 4, B. I.) the triangles $H\,I\,K$, $H\,I\,L$ are identical, making the side $L\,H$ equal to the side $H\,K$, the angle $I\,H\,L$ equal to the angle $I\,H\,K$, and the angle $I\,L\,H$ equal to the angle $I\,K\,H$; hence V is greater than the sum of the four angles $I\,H\,K$, $I\,H\,L$, $I\,K\,H$, $I\,L\,H$. If we connect the points K and L by a right line, it will intersect $I\,H$, or $I\,H$ produced, in some point, M, and the angles at M will be right angles; for the triangles $K\,H\,M$, $L\,H\,M$ are identical, since $H\,K = H\,L$, and the side $H\,M$ common, and that the angle $K\,H\,M$ equals the angle $L\,H\,M$, $\therefore$ the angle $K\,M\,H$ equals the angle $L\,M\,H$ (Sim., p. 4, B. I.); consequently $K\,M$ is a perpendicular from the angle K of the triangle $H\,I\,K$ to the opposite side $H\,I$, or to $H\,I$ produced (Sim., def. 10, B. I.). We now observe that

KM must fall without the triangle HIK (or that it will meet HI, produced in the direction HI), for if KM does not fall without the triangle KIH, but coincides with KI, or falls at some point between H and I, then we shall have the triangle HKL such that the sum of its angles is less than V, and as V is any finite angle taken as small as we please, ∴ the sum of the angles of the triangle HKL is less than any finite angle, or it is infinitesimal, which is impossible. Hence the perpendicular KM falls on HI produced in the direction HI, so as to make the angle HKM equal to some finite angle; and it is evident that the perpendicular cannot intersect HI produced in the direction IH, for if it could, a triangle would be formed having the sum of two of its angles greater than two right angles, which is impossible. Hence, since the angle HIK is the exterior angle of the triangle KIM, it is greater than the right angle KMI (Sim., p. 16, B. I.); hence the sum of the angles of any triangle is greater than a right angle, as required.

"*Prop.* 4′. The sum of the angles of any triangle is not less than two right angles.

"We shall use the figure to Prop. 3′. It is evident that we may suppose the sum of the angles of the triangle KHL not less than that of the triangle KIH, or, since the sum of the angles IHK, IKH is not greater than $\frac{A}{2^m}$, we shall have $2\,IKM + \frac{A}{2^m}$, not less than the angle KIH. Hence, if we denote a right angle by R, since the sum of the angles HIK, KIM is equal to $2R$ (Sim., p. 13, B. I.), and that the sum of the angles KIM, IKM is not greater than R (see our Prop. 2, and observe that the angle $IMK = R$), we get IKM not less than $R - \frac{A}{2^m}$, or $\frac{A}{2^m}$ is not less than the angle KIM. But $\frac{A}{2^m}$ is less than any given angle, ∴ the angle KIM is infinitesimal, consequently the angle KIH differs from $2R$ by an infinitesimal angle, and of course the sum of the angles of the triangle KIH or ABC is not less than $2R$, as required.

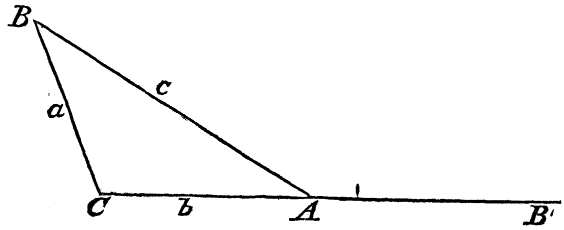
"*Cor.* Hence, since the sum of the angles of any triangle (ABC), is neither greater nor less than $2\,R$, it is equal to $2\,R$, = two right angles."

II. "*An attempt to show* (*analytically*) *that the sum of the angles of any rectilineal triangle is equal to two right angles.*

"*Ax.* The angle formed by two (right) lines is independent of the lengths of the lines.

" *Prop.* 1. To express any side of a triangle in terms of the other sides and their included angle.

" Let ABC be any triangle; and suppose its sides BC, AC, AB severally contain some assumed length (considered as the unit of



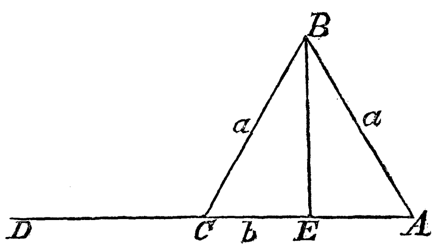
length) a, b, c times, then will the sides be expressed by a, b, c; where it may be observed that a, b, c are positive, and that they may be integral or fractional, rational or surd, according to the nature of the case; we shall denote the angle BAC by A, and shall suppose CA to be produced (in the direction CA) to B', so that $AB' = BA$, then (Simson's Euclid, Book I., prop. 13) the angle BAB' is the supplement of A, or the sum of A and BAB' is equal to two right angles.

"By Sim., B. I., p. 20, we have the inequalities $a + b > c$, $a + c > b$, or (which is equivalent to them), we have $a > \pm (c - b)$, (1); in which we must use the upper sign when c is greater than b, and the lower sign must be taken when c is less than b; and it is manifest that (1) exists even when $c = b$. In order to remove the ambiguous sign, we may (by taking the second power of a, and $\pm (c - b)$ put (1) under the form $a^2 > (c - b)^2$, or $a^2 - (c - b)^2 > 0$, (2). If the angle $A = 0$, AB falls on AC, and (2) evidently becomes $a^2 - (c - b)^2 = 0$, which is its least value; and, Sim., B. I., p. 24, if we suppose b and c each invariable, and the angle A to be increased, then A will be increased, and the greatest value that a can have will be when the angle A equals two right angles, or when AB coincides with AB', and $a = b + c$, so that (2) becomes $(b + c)^2 - (b - c)^2 = 4bc$, which is its greatest value; hence and by (2) if we put $\frac{a^2 - (c-b)^2}{2bc} = p$, (3), p cannot be less than 0 (or cannot be negative), nor greater than 2, or p has 0 for its lesser, and 2 for its greater limit. From (3) we get $a^2 = (c - b)^2 + 2pbc = b^2 + c^2 - 2(1 - p)bc$, or if we put $1 - p = n$, (4), then $a^2 = b^2 + c^2 - 2nbc$, (5); where, since p never passes the limits 0 and 2,

it is evident by (4) that n cannot pass the limits $+1$ and -1, and that n depends on the angle A; also that $n=1$ corresponds to $A=0$, and $n=-1$ to $A=$ two right angles; so that a is expressed in (5) as required.

"*Prop.* 2. To find the value of n, that corresponds to the base-angles of any isosceles triangle.

"Let $A B C$ be any isosceles triangle; having $A B = CB = a$, for its sides, and $A C = b$ for its base, and let the base be produced in the direction $A C$ to any point, D, then, Sim., B. I., p. 13, the angle $B C D$ is the supplement of the angle $B C A$. Bisect the base of the triangle in E, then draw the right line BE from the vertex B to E, and the triangles $A B E$, CBE are identical (Sim., B. I., pp. 8 and 4); so that the angle $A E B$ equals the angle $C E B$, and these angles are right, Sim., B. I., def. 10, and BE is perpendicular to the base of the triangle. By (5) of prop. 1, we get $a^2 = a^2 + b^2 - 2nba$, or by reduc. $b = 2na$, or $n = \frac{\frac{b}{2}}{a} = \frac{AE}{AB} = \frac{CE}{CB}$, as required. Also, if we use m instead of n, for the vertical angle (B), we have $b^2 = a^2 + a^2 - 2maa = 2(1-m)a^2$, or $1 - m = \frac{b^2}{2a^2}$, or since $b = 2na$, we get $\frac{b^2}{2a^2} = 2n^2$, $\therefore 2n^2 = 1 - m$, or $n = \pm\sqrt{\frac{1-m}{2}}$, (1), which is another form of n; and it is manifest that if we take the upper sign before the radical for the value of n that corresponds to the acute angle $B C A$, we must take the lower sign before the radical in order to get the value of n that corresponds to the obtuse angle $B C D$, which (as before noticed) is the supplement of $B C A$ ($= B A C$).

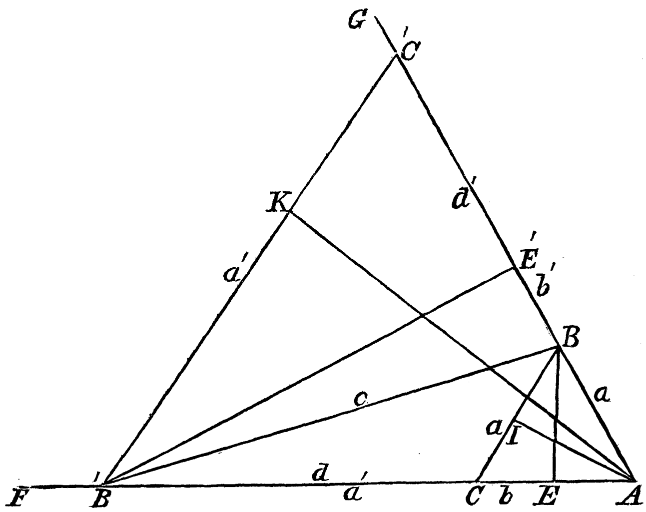


"*Cor.* 1. By what has been done it is evident, that, if we divide one of the legs of a right-angled triangle by the hypothenuse, we get the value of n that corresponds to the included angle; for evidently the same value of n corresponds to the isosceles triangle $A B C$, and to the identical right triangles into which it is divided by the perpendicular $B E$ from its vertical angle.

"*Cor.* 2. It is manifest from (1), that all those isosceles triangles which have equal values of n for their base angles also have equal values of m for their vertical angles.

"*Prop.* 3. All right-angled triangles which have one of their acute angles common or equal will have equal values of n corresponding to the common or equal angles.

"Let $A\,B\,E$, $A\,B'E'$ be two right triangles, right-angled at E and E', and having the common angle A the hypothenuse of the one



and leg of the other (which include the common angle A) being in the right lines $A\,G$, $A\,F$, which include the angle A, that is common to the two triangles; that is, $A\,B$ and $A\,E'$ are in $A\,G$, $A\,B'$ and $A\,E$ in $A\,F$. When the angles are equal, but not common, we may imagine $A\,B\,E$ to be one of the triangles, and we may suppose the leg of the other triangle that is adjacent to the angle that is equal to A to be applied to $A\,G$, so that the angle which equals A shall coincide with A; then will the hypothenuse of the applied triangle lie on $A\,F$, and we may conceive that $A\,B'\,E'$ represents the applied triangle; so that the case of equal angles is reduced to that of a common angle. Let $E\,C = A\,E$ from E towards F, and $E'C' = A\,E'$ from E' towards G, then draw right lines from C to B, and from C' to B'; and it is evident by Sim., B. I., p. 4, that $A\,B\,C$, $A\,B'C'$ are isosceles triangles, $B\,E$, $B'E'$ being the perpendiculars from their vertical angles to their bases. Join the vertices of the isosceles triangles by the right line $B\,B' = c$, and put $A\,B = B\,C = a$, $A\,C = b$, $\frac{AE}{AB} = \frac{CE}{CB} = \frac{\frac{b}{2}}{a} = n$; also put $A\,B' = B'\,C' = a'$, $A\,C' = b'$,

$\frac{AE'}{AB'} = \frac{CE'}{C'B'} = \frac{\frac{b'}{2}}{a'} = n'$; also let $C B' = d$, $C' B = d'$. From the triangle $A B B'$ we get, by prop. 1 (by using N, instead of n, to represent the angle A in this triangle, since n represents the angle A in the isosceles triangle $A B C$), $c^2 = a^2 + a'^2 - 2 N a a'$, (1), or if we put $N = n x$, we get $c^2 = a^2 + a'^2 - 2 n x a a'$; and in like manner we get from the triangle $B' B C$, $c^2 = a^2 + d^2 \pm 2 n x' a d$, (2); where for $\pm$ we must use $+$ when B' is not between the points A and C, and $-$ must be used for $\pm$ when B' is between A and C, as is evident from (1) of prop. 2. Equating the two values of c^2, we get, after a slight reduction, $a'^2 - d^2 - 2 n x a a' \mp 2 n x' a d = 0$, (3); since $2 n a = A C$, and $a'^2 - d^2 = (a' + d)(a' - d) = (a' \pm d) \times A C$ (the upper sign being used when B' is not between A and C, and $-$ in the contrary case); hence, substituting the values of $2 n a$ and $a'^2 - d^2$, by rejecting the common factor $A C$, (3) is reduced to $a' \pm d - x a' \mp x' d = 0$, or $a'(1 - x) \pm d(1 - x') = 0$, or since $a' = b \pm d = A C \pm C B'$ (using the upper sign when B' is not between A and C, and $-$ when it is between A and C), we get $A C(1 - x) \pm C B'(1 - x + 1 - x') = 0$, (4). Now it is evident that $C B'$ must be arbitrary, and not dependent on $A C$ or $1 - x$, $1 - x'$; $\therefore$ we must have $1 - x + 1 - x' = 0$, and (4) is reduced to $A C(1 - x) = 0$, which, since $A C$ is not $= 0$, gives $1 - x = 0$, $\therefore$ $1 - x' = 0$, or $x = 1$, $x' = 1$; hence (1) and (2) become $c^2 = a^2 + a'^2 - 2 n a a'$, (1'), $c^2 = a^2 + d^2 \pm 2 n a d$, (2'). In like manner, by regarding the angle A as belonging to the isosceles triangle $A B' C'$, we get from the triangles $A B B'$, $B C' B'$, $c^2 = a^2 + a'^2 - 2 n' a a'$, (1''), $c^2 = a'^2 + d'^2 \pm 2 n' a' d'$, (2''); where for $\pm$ we must use $-$ when B is between A and C', and $+$ when B is in $A C'$ produced beyond C'. By equating the values of c^2, as given by (1') and (1''), we get $n' = n$, or $\frac{AE'}{AB'} = \frac{AE}{AB}$, as was to be proved. It is evident that $A B E$ may represent any right triangle having A for one of its acute angles, and its hypothenuse on $A G$; also $A B' E'$ may denote any right triangle which has A for one of its acute angles, and its hypothenuse on $A F$; hence, from what has been shown, n will be the same for all the triangles represented by $A B E$ and $A B' E'$; that is, all right triangles which have a common or equal acute angle will have equal values of n corresponding to the common or equal acute angle. There is one case that apparently forms an exception to what has been shown; and that is when the hypothenuse of a tri-

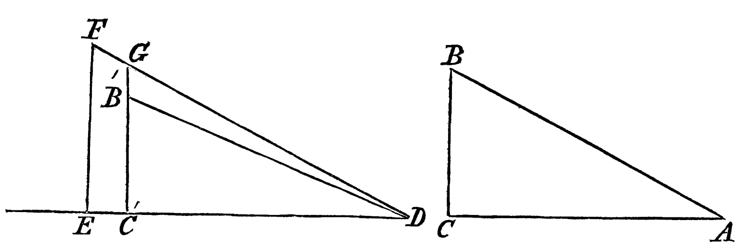
angle that lies in one of the lines AF, AG coincides with the leg of another triangle that lies in the other of these two lines; but this exception is only apparent, for the value of n in these two triangles is the same as that of n in the two triangles ABE, $AB'E'$, $\therefore$ when the hypothenuse of one triangle coincides with a leg of another triangle, the value of n, that corresponds to A in one of the triangles, is the same as in the other triangle.

" *Cor.* We can now easily find the value of m that corresponds to the vertical angle of an isosceles triangle whose base-angles are represented by n (or to which n corresponds).

" For let AI be drawn from the base-angle A of the isosceles triangle ABC, perpendicular to the opposite side BC, meeting it in I, then from what has been shown we get $CI = nb$, or (since $b = 2na$) $CI = 2n^2a$, $\therefore BI = a - CI = a(1 - 2n^2)$, and (since by (1) of prop. 2, $m = 1 - 2n^2$) we get $m = \frac{BI}{a} = \frac{BI}{AB}$, which can also be easily obtained from other considerations. And since n corresponds equally to the base-angles of all the isosceles triangles represented by ABC, $AB'C'$, and since $m = 1 - 2n^2$, it follows that all isosceles triangles whose base-angles are equal will have equal values of m corresponding to their vertical angles.

" *Prop.* 4. If there are two right-angled triangles, such that a leg of the one divided by its hypothenuse gives the same quotient as a leg of the other divided by its hypothenuse, then shall the angle included by the leg and hypothenuse of the one triangle be equal to the angle included by the leg and hypothenuse of the other triangle.

" Let ABC, DFE be two triangles right-angled at C and E, such that $\frac{AC}{AB} = \frac{DE}{DF} = n$; then shall the angle BAC equal the angle



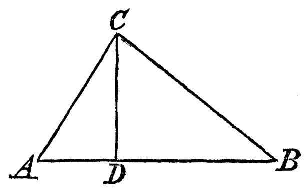
FDE. For on the longer leg DE of the one take DC' equal to AC, and through C' draw $C'G$ perpendicular to DE, meeting DF in G, then shall the triangles ABC, DGC' be identical; for since the right triangles DFE, DGC' have the angle D common, we

have, by prop. 3, $\frac{DC'}{DG} = \frac{DE}{DF} = n$, $\therefore$ $\frac{DC'}{DG} = \frac{AC}{AB}$, and since $D C' = A C$, we get $D G = A B$. If we take $C'B' = CB$, and draw a right line from D to B', the triangles $A B C$, $D B'C'$ are identical, Sim., B. I., p. 4, $\therefore$ $D B' = A B$, and of course $D B' = D G$, which cannot be unless B' falls on G; for of the base-angles $D B'G$, $D G B'$, one is acute and the other obtuse, and the same holds true whether B' is within or without the triangle $D F E$; hence we cannot have $D B' = D G$ unless B' coincides with G, Sim., B. I., p. 19. Hence, since the triangles $A B C$, $D B'C'$ are identical, and that B' falls on G, the triangles $A B C$, $D G C'$ are identical, and the angles $B A C$, $F D E$ are equal, as required.

" *Cor.* If we draw $A K$ at right angles to $B'C'$, meeting it in K (see fig., prop. 3), then since the base-angles of the isosceles triangles represented by $A B C$, $A B'C'$ are equal, we have, by cor. to prop. 3, $m = \frac{BI}{AB} = \frac{B'K}{AB'}$, and hence the vertical angles of the isosceles triangles are equal; and the same holds true whether the perpendiculars $A I$, $A K$ fall within the triangles (as in the figure) or without them; for when the perpendiculars fall without the triangles, the equality $\frac{BI}{AB} = \frac{B'K}{AB'}$ shows that the supplements of the vertical angles of the triangles are equal, and consequently the vertical angles are equal; and it is evident, by cor. to prop. 3, that the perpendiculars will both at the same time fall within or without the triangles; the case when $m = 0$, or $B I = 0$, $B'K = 0$, is too evident to require any explanation, for the vertical angles are evidently right. Hence the angles $A B E$, $A B'E'$, the halves of the vertical angles of the isosceles triangles, are equal; *hence it follows that all those right-angled triangles which have an acute angle common, or equal, have their other acute angles also equal.* Hence (see fig. 3) from the right triangles $A B E$, $A B'E'$, having their angles B, B' equal, we get $\frac{BE}{AB} = \frac{B'E'}{AB'}$, and since $\frac{AE}{AB} = \frac{AE'}{AB'}$ we deduce $\frac{BE}{AE} = \frac{B'E'}{AE'}$; that is, if we have two (or more) right triangles which have an acute angle common or equal, then if we divide the leg of any one of them which is opposite to the (equal) angle by the leg adjacent to the angle, the quotient will equal the corresponding quotient obtained in like manner from any other (one) of the triangles; and the converse of what is here affirmed is also easily proved to be true in a manner very analogous to that given in proving the proposition above.

" *Prop.* 5. The sum of the acute angles of a right triangle equals a right angle, and the sum of the squares of the legs equals the square of the hypothenuse.

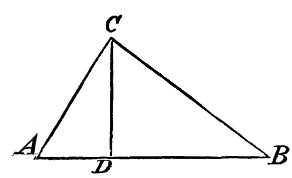
"Let ACB be the triangle, having the angle C right; from C draw CD, at right angles to the hypothenuse, meeting it at D; hence cor., prop. 4, since the right triangles ACD, ACB have the angle A common, their other acute angles ACD and B are equal; also since the (right) triangles BCD, ABC have a common angle, B, their other acute angles BCD and A are equal. Hence the sum of the angles A and B is equal to the sum of the angles BCD and ACD, which compose the right angle ACB, and of course the sum of the angles A and B is equal to a right angle; and consequently the sum of all the angles of the triangle ACB is equal to two right angles. Again, the right triangles ACB, ACD having the common angle A, by prop. 3, give the equality $\frac{AC}{AB} = \frac{AD}{AC}$, or $AC^2 = AB \, . \, AD$; and in the same way we get from the triangles ACB, BCD, $BC^2 = AB \, . \, BD$; and consequently $AC^2 + BC^2 = AB^2$, as required.



"*Cor.* Since the right triangles ACD, BCD have the angles CAD, BCD equal, they (by the cor. to prop. 4) give the equality $\frac{CD}{AD} = \frac{BD}{CD}$, or $CD^2 = AD \, . \, BD$.

"*Prop.* 6. The sum of the angles of any triangle is equal to two right angles.

"Let ACB denote any triangle; and suppose that the angle C is not less than either of the other angles of the triangle, and that the perpendicular CD is drawn from C to the opposite side AB; then, Sim., p. 17, B. I., CD will fall within the triangle ACB. Hence, since the triangles ACD, BCD are right-angled at D, by the last prop. the sum of the acute angles A and ACD of the first of these triangles is equal to a right angle; and in the same way the sum of the acute angles B and BCD of the second triangle is equal to a right angle; but the sum of the acute angles of these triangles equals the sum of the angles of the triangle ACB; consequently, the sum of the angles of the triangle ACB is equal to two right angles, as required.



"In conclusion, we will remark that the relation of what has been

done to the doctrine of similar triangles and the science of trigonometry is too evident to require any comment."

Two hundred and ninety-fifth Meeting.

May 4, 1847. — MONTHLY MEETING.

The VICE-PRESIDENT in the chair.

Professor Peirce announced that he had continued and nearly completed his researches into the irregularities of motion exhibited by Uranus, and was more strongly than ever of the opinion that they were not to be attributed to the influence of the newly discovered planet Neptune. He had obtained several possible solutions of the problem, which are different from those of Leverrier and Adams, and which are published in a communication to the *Boston Courier*, dated April 29, 1847, and which he now proposes to lay before the Academy.

" The problem of the perturbations of Uranus admits of three solutions, which are decidedly different from each other, and from those of Leverrier and Adams, and equally complete with theirs. The present place of the theoretical planet, which might have caused the observed irregularities in the motions of Uranus, would, in two of them, be about *one hundred and twenty degrees* from that of Neptune, the one being behind, and the other before, this planet. If the above geometers had fallen upon either of these solutions instead of that which was obtained, Neptune would not have been discovered in consequence of geometrical prediction. The following are the approximate elements for the three solutions at the epoch of Jan. 1, 1847.

	I.	II.	III.
Mean Longitude, . . .	319°	79°	199°
Longitude of Perihelion, . .	148	219	188
Eccentricity, . . .	0.12	0.07	0.16

In each of them (the mass of the sun being unity)

The mass is 0.0001187

" The period of sidereal revolution is double that of Uranus. It will be observed that the mean distance in all these cases is the same with that of Neptune, and that, in the first* of them, the present direction

* The first of these solutions is corrected from the one which was published in